Dirichlet problem and harmonic measure

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<u>Dirichlet problem</u>: Ω – domain, $f \in C(\partial\Omega)$. Find: $u \in C(\bar{\Omega})$, $u|\partial\Omega = \varphi$, $\Delta u = 0$ in Ω

If solvable, $z_0 \in \Omega \Rightarrow$ $\varphi \to u(z_0)$ – positive functional on $C(\bar\Omega)$, with norm 1.

By Riesz, $\exists\, \omega_{z_0 \Omega}$: $u(z_0) = \int \varphi \, d\omega_{z_0,\Omega}$ $z_0 \to \omega_{z_0}$ – harmonic

By Harnack, $\omega_{z_1} \asymp \omega_{z_2}$.

In disk, Dirichlet problem is solvable by Poisson:

$u(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\xi|^2} \varphi(\xi) \, d|\xi|$.

So, in the locally connected case, since $u \in \mathrm{Harm}(\Omega)$, $f: \mathbb{D} \to \Omega$ – conf.

$u \circ f \in \mathrm{Harm}(\mathbb{D})$, we get that $u(w) = \int_{\mathbb{T}} \frac{1-|f^{-1}(w)|^2}{|f^{-1}(w) - \xi|^2} \varphi \circ f(\xi) \, d|\xi|$.

In particular, if $f(0) = w$, we get

$\omega_{z_0,\Omega} = f_* \Lambda$, Λ – linear measure on $\mathbb{T}$.

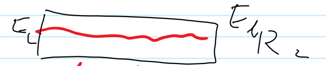
For general s.c. Ω – still solvable and $\omega_{z_0,\Omega} = \hat{f}_* \Lambda$, but requires potential Theory, so we'll skip the proof. Notion: $\omega(z_0, K, \Omega) := \omega_{z_0,\Omega}(K)$

ω is naturally defined on $\hat\partial(\Omega)$ – Carathéodory boundary by $\hat f(\Lambda)$.

In particular, in the upper half-plane $\mathbb{H}$:

$\omega(z, \cdot, \mathbb{H}) = \frac{1}{\pi} \frac{\operatorname{Im} z}{|t-z|^2} dt$, and $\omega(z, [a,b], \mathbb{H}) = \arg\left(\frac{z-b}{z-a}\right)$.

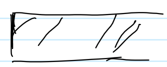
<u>Harmonic measure in a rectangle:</u>



Let $R_L := \{|\operatorname{Re} z| < L, |\operatorname{Im} z| < 1\}$. If Γ = curve family joining vertical sides, $\lambda(\Gamma) = L$.

$E_L := \{z \in \partial R_L : |\operatorname{Re} z| = L$

<u>Lemma</u>. $e^{-\frac{\pi}{2}L} \le \omega(0, E_L, R_L) \le \frac{8}{\pi} e^{-\frac{\pi}{2}L}$ $\omega(0, E_L, R_L) e^{\frac{\pi}{2}L} \nearrow 1$ as $L \to 0$, $\searrow \frac{8}{\pi}$ as $L \to \infty$.

<u>Pf</u>. Let  $S_L := \{|\operatorname{Im} z| < 1, \operatorname{Re} z > -L\}$.

Let $u(z) := \omega(z, \operatorname{Im} z = -L, S_L)$. Then $\omega(z, E_L, R_L) \ge u(z)$.

On the other hand, $\omega(z, \operatorname{Im} z = -L, S_L) = \frac{1}{2}\omega(z, E_L, R_L) \le u(z)$.

Map $z \to e^{\pi z/2}$ maps S_L to $H_L := \{|w| > e^{-\pi L/2}, \operatorname{Im} w > 0\}$.

Then $\omega(w, |w| = e^{-\pi L/2}, H_L) = \frac{2}{\pi} \arg\left(\frac{w + ie^{-\pi L/2}}{w - ie^{-\pi L/2}}\right)$ 2.10

$u(0) = \omega(1, |w| = e^{-\pi \frac{L}{2}}, H_L) = \frac{4}{\pi} \arctan(e^{-\frac{\pi}{2}L})$.

Now, observe that $\frac{\pi}{4}t \le \arctan t \le \min(t, \frac{\pi}{4})$, with = almost reached asymptotically when $t \to 1, 0$ respectively.

<u>Thm</u>: Let Ω – s.c, $E \subset \hat\partial(\Omega)$: $\hat f^{-1}(E)$ – arc on $\mathbb{T}$.

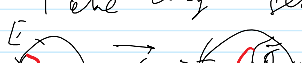
For $z_0 \in \Omega$, define $\lambda(z_0, E) = \sup \lambda(\Gamma)$, where Γ are curve families connecting some crosscut σ from z_0 to E.

<u>Then</u>: $e^{-\pi\lambda(z_0,E)} \le \omega(z_0, E, \Omega) \le \frac{8}{\pi} e^{-\pi\lambda(z_0,E)}$.

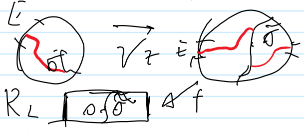
<u>Pf</u>. Everything is conformally invariant, so, can assume:

E – arc of $\mathbb{T}$, $z_0 = 0$, $\Omega = \mathbb{D}$, $|E| = \omega(0, E, \mathbb{D})$.

Take any semicrosscut σ from 0 to $\mathbb{T}$. Apply $\sqrt{z}$. σ is mapped to

 some crosscut $\tilde\sigma$, $\mathbb{D} \setminus \sigma$ mapped by two branches of

Take any semicrosscut σ from 0 to $\mathbb{T}$: Apply $\sqrt{z}$. σ is mapped to some crosscut $\tilde{\sigma}$, $\mathbb{D}\setminus\sigma$ mapped by two branches of $\sqrt{z}$ to $\mathbb{D}\setminus\tilde{\sigma}$. E is mapped to two arcs E_1, E_2, $|E_1| = |E_2| = \frac{1}{2}|E|$.



$\exists f$: conformal map of $(\mathbb{D}, E_1, E_2)$ to some conformal rectangle R_L, $f(0)=0$ (symmetry), with L such that $\omega(0, E_L, R_L) = |E| \,(= |E_1| + |E_2|)$.

Let us look at the extremal length: Γ is mapped by two branches of $f\circ\sqrt{z}$ to two curve families Γ_1, Γ_2, each connecting corresponding side of R_L to $\tilde{\tilde{\sigma}} := f(\tilde{\sigma})$. $\lambda(\Gamma_1) = \lambda(\Gamma_2) = \lambda(\Gamma)$, but $\lambda(\Gamma_1) + \lambda(\Gamma_2) \le L$, by serial rule. Equality is reached when $\tilde{\tilde{\sigma}}$ is vertical line, i.e. when σ is a radius dividing $\mathbb{T}\setminus E$ into two equal halfs!

Thus $\lambda(\Gamma) \le \frac{L}{2}$, with $\lambda(z_0, E) = \frac{L}{2}$. Now use Lemma! ∎

Thm If E is a finite union of arcs in $\partial(\Omega)$, then

$$\omega(z_0, E, \Omega) \le \frac{8}{\pi} e^{-\pi \lambda(\Gamma)},$$

where σ - any semicrosscut from z_0 to $\partial(\Omega)\setminus E$, Γ - curve family joining E to σ.

Pf. Similar to previous, but need to map $\mathbb{D}\setminus\sigma$ to complement of slit domain, use $\rho \equiv 1$ for the dual metric. Details: in G-M. ∎

Corollary. Let $w \in \partial\Omega$, $w_0 \in \Omega$, $\mathrm{dist}(w_0, w) > \frac{1}{2}$, $r_0 < \frac{1}{2}$.

Then $\omega_\Omega(B(w, r_0) \cap \partial\Omega, w_0) \le \frac{8}{\pi} \exp\left(-\pi \int_{r_0}^{1/2} \frac{dr}{r\theta(r)}\right)$ where $\theta(r)$ is the angular measure of $\Omega \cap \{z : |z - w| = r\}$. Remark. $\frac{1}{2}$ can be replaced by anything.

Pf.

First, observe that if we prove the estimate for $\Omega_\eta = f(\eta\mathbb{D})$, where $\eta < 1$, $f: \mathbb{D} \to \Omega$ conformal, then we can pass to the limit when $\eta \to 1-$. But for Ω_η, $B(w, r_0) \cap \partial\Omega_\eta$ is a union of finitely many arcs. So we can apply previous Thm with some semicrosscut in Ω from w_0 to $\partial\Omega$, not intersecting $B(w, \frac{1}{2})$.

So $\omega_{\Omega_\eta}(B(w, r_0) \cap \partial\Omega_\eta, w_0) \le \frac{8}{\pi} \exp(-\pi\lambda(\Gamma))$

But $\lambda(\Gamma) \ge \lambda(\tilde{\Gamma})$ where $\tilde{\Gamma}$ are curves in Ω_ρ joining $\{|z-w| = r_0\}$ with $\{|z - w| = \frac{1}{2}\}$. Consider $\rho(z) = \begin{cases} \frac{1}{r\theta(r)} & \text{if } |z-w| = r,\ z \in \Omega,\ r_0 \le r \le \frac{1}{2} \\ 0, & \text{other } z. \end{cases}$

Then $L_\rho(\tilde{\Gamma}) \ge \int_{r_0}^{1/2} \frac{dr}{r\theta(r)}$, $A(\rho) = \iint \frac{r\,dr}{r^2(\theta(r))^2} d\theta\, dr = \int_{r_0}^{1/2} \frac{dr}{r\theta(r)}$,

so $\lambda(\Gamma) \ge \lambda(\tilde{\Gamma}) \ge \int_{r_0}^{1/2} \frac{dr}{r\theta(r)}$ ∎

Some corollaries.

Corollary 1 (Beurling estimate).

$\omega(B(z, r_0)) \le C r_0^{1/2}$ if $z \in \partial\Omega$

Pf. $\theta(r) \le 2\pi$, so $\omega(B(z, r)) \le \frac{8}{\pi}\left(\exp\left(-\pi \int_{r_0}^{1/2} \frac{dr}{r\, 2\pi}\right)\right) =$

$\frac{8}{\pi} \exp\left(\frac{1}{2}\left(\log r_0 - \log\frac{1}{2}\right)\right) = \frac{8\sqrt{2}}{\pi} r_0^{1/2}$, if $\mathrm{dist}(w_0, \partial\Omega) > \frac{1}{2}$. Scale ∎

Remark. Can also be obtained from Lavrentiev.

Corollary 2. Let γ - Jordan curve, Ω_-, Ω_+ - two domains with boundary γ, $w_0 \in \Omega_+$, $\mathrm{dist}(w_0, \gamma) > \frac{1}{2}$, $w \in \gamma$.

$\omega(B(w,r_0),\Omega_+,a)\,\omega(B(w,r_0),\Omega_-,\infty)\le Cr^2.$
$\quad\ \ \omega_+ \qquad\qquad\qquad\qquad \omega_-$

Pf. $\omega_+ \le \frac{8}{\pi}\exp\left(-\pi\int_{r_0}^{r/2}\frac{dr}{r\,\theta_+(r)}\right)$. But $\theta_+ + \theta_- \le 2\pi$, so

$\omega_+\omega_- \le \left(\frac{8}{\pi}\right)^2 \exp\left(-\pi\int_{r_0}^{r/2}\frac{dr}{r}\left(\frac{1}{\theta_+(r)}+\frac{1}{\theta_-(r)}\right)\right) \le \left(\frac{8}{\pi}\right)^2 \exp\left(-\pi\int_{r_0}^{r}\frac{2}{\pi}\frac{dr}{r}\right)$

$\le \frac{2^8}{\pi^2}\,r^2.$ ∎